On cutting planes for convex mixed-integer programs

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1 Introduction

Let $n, p, q \in \mathbb{Z}_+$ with $n = p + q$. Let $c \in \mathbb{R}^p, d \in \mathbb{R}^q$ and let $K \subseteq \mathbb{R}^n$ be a convex set. A convex mixed integer program (CMIP) is an optimization problem of the form

$$\inf\{c^T x + d^T y \,|\, (x, y) \in K \cap (\mathbb{Z}^p \times \mathbb{R}^q)\}. \tag{1}$$

A *cutting plane* for K is a linear inequality $\alpha x + \beta y \leq \gamma$ that is satisfied by all $(x, y) \in K \cap (\mathbb{Z}^p \times \mathbb{R}^q)$. Let $\mathcal{F}$ be a collection of cutting planes. A cutting plane algorithm to solve (1) that uses cuts in $\mathcal{F}$ can be described as follows:

Cutting plane Algorithm.

1. Solve $\inf\{c^T x + d^T y \,|\, (x, y) \in K\}$. If the optimal solution (x^*, y^*) satisfies $x^* \in \mathbb{Z}^p$, stop.
2. Otherwise, find a cutting plane $\alpha x + \beta y \leq \gamma$ in $\mathcal{F}$ such that $\alpha x^* + \beta y^* > \gamma$. Redefine $K := \{(x, y) \in K \,|\, \alpha x + \beta y \leq \gamma\}$. Repeat.

Some conditions on the family $\mathcal{F}$ that are necessary to obtain a well-defined algorithm are: (i) The cutting plane in the second step must exist. (ii) The procedure should be able to terminate in a finite number of steps. Therefore, it is of interest to answer the following question:

Are there non-trivial families of cutting planes that satisfy conditions (i) and (ii)? $(*)$

For pure integer problems $(p > 0, q = 0)$ question $(*)$ has been answered affirmatively in the case the set K is a rational polyhedron or a compact convex set: the family of Chvátal-Gomory cuts satisfies conditions (i) and (ii) (see [6, 15, 19]). In the rest of this document we will focus our attention on the mixed-integer case.

2 What is known in the Mixed-integer case $(p, q > 0)$:

In the general mixed-integer case, the family of Chvátal-Gomory cuts does not comply with (i) and (ii) (see, for example, [8]). Therefore, we need to use a larger family of cutting planes. A polyhedron $L \subseteq \mathbb{R}^n$ is said to be lattice-free if L does not contain points of $\mathbb{Z}^p \times \mathbb{R}^q$ in its relative interior. An integral lattice-free cut for K is a linear inequality valid for $\text{conv}(K \setminus \text{rel.int}(L))$, where L is a maximal integral lattice-free polyhedron.[1] The integral lattice-free closure of K is the set of points that satisfy all integral lattice-free cuts for K. Denote $\mathcal{L}^{(0)}(K) = K$. For $i \geq 1$, we define the ith integral lattice-free closure of K, denoted by $\mathcal{L}^{(i)}(K)$, as the integral lattice-free closure of $\mathcal{L}^{(i-1)}(K)$.

[1] We note here that it is known that finding cutting planes for $\text{conv}(K \setminus \text{rel.int}(L))$ can be done in polynomial time in some special cases: rational polyhedra (see [3, 4, 7]) and second order conic representable sets (see [5]).

2.1 Rational polyhedral case

Theorem 1 ([17, 12, 1])**.** *Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a rational polyhedron (that is, the matrices A and b are defined by rational numbers). Then*

1. *$\mathcal{L}^{(i)}(P)$ is a rational polyhedron for all $i \geq 0$.*
2. *There exists $t \geq 0$ such that, $\mathcal{L}^{(t)}(P) = \text{conv}(P \cap (\mathbb{Z}^p \times \mathbb{R}^q))$.*

Observe that Theorem 1 implies that the family of integral lattice-free cuts satisfies conditions (i) and (ii) in the case of rational polyhedra. For a related result see [16]. Notice that the collection of maximal integral lattice-free polyhedra is finite up to affine unimodular transformations (see [2]).

2.2 Compact convex case

In the general compact case there have been some attemps to define a family of cutting planes satisfying (i) and (ii) by using mixed-integer linear outer-approximations of the original set (see [14, 20]). However, there are some issues with this approach: First, it only works when the compact convex set is defined by convex functions that satisfy some smoothness conditions and a strong constraint qualification holds at the solution of some related non-linear programming problems. Second, the outer-approximations used to obtain cutting planes belonging to the family are not necessarilly defined by rational data.

3 Research proposal: Some extensions to compact convex sets

Our main research objective is to study whether question ($*$) has a positive answer in the case of general compact convex sets. In order to address this question, we believe that it is necessary to extend some properties of mixed-integer linear sets that are crucial in the proof of Theorem 1. We describe some of these properties next.

1. **Structural result.** When P is a rational polyhedron, as a consequence of $P_I := \text{conv}(P \cap (\mathbb{Z}^p \times \mathbb{R}^q))$ having a finite number of maximal faces, we obtain that there exists a finite collection of maximal lattice-free polyhedra $T_1, \ldots, T_N$ such that

$$P_I = \bigcap_{i=1}^{N} \text{conv}(P \setminus \text{rel.int}(T_i)). \tag{2}$$

   In the general compact convex set case, $K_I := \text{conv}(K \cap (\mathbb{Z}^p \times \mathbb{R}^q))$ may have an infinite number of maximal faces. Thus, it is not clear if a representation of K_I as in (2) is possible. Currently, we have been able to prove such a result for some special cases ($n \leq 3$ or $p = 1$).

2. **Finiteness result.** Notice that for general compact sets we cannot expect to describe $\mathcal{L}^{(1)}(K)$ with a finite number of cutting planes. However, we can ask if there exists a finite family of maximal integral lattice-free sets $L_1, \ldots, L_M$ such that

$$\mathcal{L}^{(1)}(K) = \bigcap_{i=1}^{M} \text{conv}(K \setminus \text{rel.int}(L_i)).$$

   We will base the study of this problem in some recent results about closures for general compact convex sets (Chvátal-Gomory closure [9, 10, 13, 18]; Split closure [11]).

3. **Characterization of K_I.** We would like to prove that there exists $t \geq 0$ such that, $\mathcal{L}^{(t)}(K) = K_I$.

Observe that if we are able to prove properties (1.),(2.) and (3.) we will obtain a generalization of Theorem 1 to the case of compact convex sets. We expect this result to led to the construction of well-defined algorithms for solving general convex mixed-integer programs.

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