Atomic Congestion Games with Taxes on Resources

Jiajin Yu
Mentor: Özlem Ergun

Introduction We consider the following network problem: there are finite number of users in the network. Each user has a non-negligible (or *atomic*) amount of demand and wants to route her demand from her source to her sink through multiple paths. For each edge in the network, there exists congestion for per unit of flow, which depends on the total amount of flow on this edge. The cost of a user is the total congestion she experiences for routing her demand. The social cost of the network is the sum of costs of all users. If the routing of all users can be controlled by the system, then optimizing the social cost becomes a convex optimization problem under certain natural assumptions. But it is always difficult, if not impossible, to control every user in a large system(for an example, the Internet), since the goal of a selfish user – minimizing her own cost – usually does not align with the goal of the system – optimizing the social cost. Therefore the system designer should ensure that a network performs well even when shared by selfish users.

To be more concrete, one measure of the robustness of the network can be the ratio between the social cost of the worst outcome of the stable states of the system and that of the optimal state of the system. Here the stable state can be formally defined as the Nash equilibrium of a game played by users of the network, whose strategies are the paths from their sources to their sinks. The ratio is called the Price of Anarchy (PoA). A small PoA means that the system will perform almost as well as it is centrally controlled. So the first question will be how to analyze the PoA of a game. This question has been studied by many researchers in the field of algorithmic game theory. But perhaps the more important question for the designer of the system would be: how can we make the PoA small? Specifically, the system designer should find a way to motivate the users such that their selfish behavior align with the optimal state as the system is centrally controlled. One practical way is to ask for a tax or a toll when some resource is used. In that case, the cost of a user would be the congestion she experienced plus the taxes she pays. But the system should not collect too much taxes from the users. In a real world scenario, no one will like to pay a high tax to use a system, even if the congestion of the system is very small. In summary, we hope to find the answers to the following questions: (1) How much smaller can the PoA be if taxes are allowed on resources? (2) How much taxes do the users have to pay to reach a small PoA? (3) Is there a trade-off between the PoA and the taxes?

Proposed Project and Initial Results Given a directed graph $G = (V, E)$ with k users, let each user i's goal be to route d_i units of flow from s_i to t_i. Let $\mathcal{P}_i$ be the set of paths from s_i to t_i and f_p be the amount of flow user i routes on the path $p \in \mathcal{P}_i$. For each edge $e \in E$, we have a congestion function c_e that is always nonnegative. Let f_e be the amount of flow on edge e. Assume $f_e \cdot c_e(f_e)$ is convex for every $e \in E$. A strategy of user i is the amount of flow she will route on each possible path $p \in \mathcal{P}_i$. The user i has cost $C_i = \sum_{p \in \mathcal{P}_i} f_p \sum_{e \in p} c_e(f_e)$. The cost of the system is $SC(f) = \sum_i C_i = \sum_{e \in E} f_e c_e(f_e)$. A pure Nash equilibrium of the game is defined as a flow where no user can decrease her cost by changing her strategy unilaterally.

The following *local smoothness* framework was introduced by Roughgarden and Schoppmann [6] to analyze the PoA of the atomic splittable congestion game. For a cost-minimization game, it is locally

(λ, μ)-smooth with respect to the outcome y if for every outcome x we have

$$\sum_{i=1}^{n}(C_i(x) + \nabla_i C_i(x)(y^i - x^i)) \leq \lambda \cdot SC(y) + \mu \cdot SC(x).$$

Here $\nabla_i C_i := (\partial C_i/\partial x_1^i, \ldots, \partial C_i/\partial x_m^i$ which denotes the gradient of C_i with respect to user i's strategy x^i. The PoA of a (λ, μ)-smooth game is $\lambda/(1-\mu)$. In [6], they showed the PoA of the atomic splittable game is $\left(\frac{1+\sqrt{d+1}}{2}\right)^{d+1}$ when the congestion functions are polynomials with degree at most d.

We set tax $t_e = \alpha \cdot c_e(f_e^*)$ where $\alpha = \alpha(d)$ is a constant depending on d and f_e^* is the optimal flow on edge e, and we are able to show that the PoA can be reduced to $\left(\frac{1+\sqrt{d+1}}{2}\right)^{d}$ (i.e., shaving a $O(\sqrt{d})$ factor) by using the local smoothness framework.

But we believe much further work can be done here, because only local information of the optimal solution is used to set taxes in the result above. To better understand how global information of the optimal can help us to set taxes to induce a game of a small PoA, we first focus on the symmetric game setting where each agent has the same d_i and s_i, t_i. In such setting, we have some initial results showing that using global information of the optimal can be beneficial for both reducing the PoA and minimizing the amount of taxes. In particular, we proved that there exists a set of taxes to induce the optimal flow as the unique Nash equilibrium of the game. For such kind of taxes, we obtain the following characterization of the taxes that minimizes $\sum_e f_e t_e$. For each path P from s_i to t_i, $\sum_{e\in P} t_e = \frac{k-1}{k}\left(\sum_{e\in P_1} c_e(f_e) - \sum_{e\in P} c_e(f_e)\right)$ where P_1 is a path in the optimal solution that has the largest marginal cost $\sum_{e\in P}(c_e(f_e) + f_e c_e'(f_e))$. Through the characterization, we are able to get an LP for optimizing the total amount of taxes $\sum_e t_e f_e$, whose dual also has a very natural interpretation. Based on that, we want to get a combinatorial algorithm to find the taxes. We think some properties of the taxes can be found during the combinatorial algorithm. By generalizing such properties of taxes properly, we hope to further reduce the PoA of the atomic congestion game and keep the amount of taxes small in the mean time.

The previous work of setting taxes to reduce PoA in the atomic splittable congestion game that we know of is the one by Cominett et.al [3]. They first considered the game without taxes and gave the PoA for games with polynomial congestions functions of degree at most 3. Then based on that they showed certain taxes can reduce the PoA. We improved their results. When the congestion functions are linear, we find taxes that induce the PoA to 5/4, smaller than the 4/3 shown in [3]. We are also able to find taxes to reduce the PoA when congestion functions are polynomials with degree at most d for general d, while in [3] the degree $d \leq 3$. The only work of inducing optimal flow as Nash equilibrium and optimizing taxes is the paper by Swamy [7], in which he showed that there exist taxes that induce a Nash equilibrium that is the same as the optimal and linear functions of such taxes can be optimized. But since multiple Nash equilibria may exist[1], the result said nothing about the PoA of the game given such taxes nor the amount of taxes to make the PoA small.

Extension We also want to consider the case of the unsplittable flow where each user can only choose one path from $\mathcal{P}_i$. We want to attack the problem again using the smooth game framework introduced by Roughgarden[5]. Caragiannis et al. [2] consider the cases when congestion functions are linear. Their work uses thes mooth analysis implicitly. Therefore we think the same technique can be applied to more general congestion functions (eg., all polynomial functions).

Another extension would be to find a set of taxes that ensures the existence of pure Nash equilibrium in the weighted unsplittable congestion game, where pure Nash equilibrium does not necessarily exist. The existence of pure Nash equilibrium is important as it is difficult to interpret the mixed Nash equilibrium in real world scenarios. In [4], a new way of assigning costs to players is proposed to ensure the existence of pure Nash equilibrium in the game. Whether certain taxes on edges can ensure the existence of pure Nash equilibrium is still an open question. Similarly, in the splittable case we may consider whether there exists a set of taxes that ensures the uniqueness of Nash equilibrium.

References

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